Simple Analytic Formulas for Dielectric Waveguides

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Abstract

We present here a set of simple analytic formulas for determining the propagation characteristics of multilayer dielectric structures. These formulas are particularly useful for computer-aided design of dielectric waveguides.

I. Introduction

Dielectric waveguides have been investigated for the design of circuit components and devices at high frequencies, particularly at millimeter wave and optical frequencies. Basic wave phenomena associated with this class of waveguides have been well known and extensive numerical data are available in the literature[1-4]. However, because of mathematical difficulties in dealing with transcendental dispersion relations, the design of dielectric waveguides has to rely on numerical procedures. Such an approach is time consuming and does not provide direct physical understanding of wave phenomena involved. The recent resurgence of interest in integrated optics has pushed the need of dielectric waveguides in a large scale, and efficient algorithms for the development of software for computer-aided design are in demand. In this work we present a set of simple analytic formulas for planar, multilayer dielectric waveguides. Such a class of structure is of practical importance on its own right; more importantly, it also serves as a building block for the analysis more complicated waveguide structures.

A single dielectric layer in an uniform environment is the simplest structure to be used as a dielectric waveguide. We observed that the dispersion relations for both TE and TM modes suported by this simple structure can be cast in a new form that is particularly simple and useful for mathematical and physical interpretations of the surface waves. Therefore, we choose the formulation of this simple structure as a building block for the analysis of more general dielectric waveguides. This new method of analysis is briefly reviewed first. The results are then applied to the analysis and design of practically interesting components and devices. For simplicity, we restrict ourself here to the fundamental TE mode; the results for the other modes are similar but slightly more complicated.

II. New Method of Analysis

Consider a dielectric waveguide consisting of a single dielectric surrounded by a uniform medium, as shown in Fig. 1. The layer has a dielectric constant ϵ_f and thickness t_f and the surrounding has a dielectric constant ϵ_a. The dielectric constants, ϵ_f and ϵ_a, are both real and positive, with $\epsilon_f > \epsilon_a$. Furhtermore, the time variation of the form, $\exp(j\omega t)$, is implicitly assumed throughout this work.

The dispersion relations for such a waveguide structure are well known; they are usually expressed in terms of the tangent or cotangent functions. For the fundamental TE mode, we have:

$$q = p \tan(p) \tag{1}$$

where the variables p and q are defined as:

$$p = k_o(t_f/2)(\epsilon_f - \epsilon_{eff})^{1/2} \tag{2}$$

$$q = k_o(t_f/2)(\epsilon_{eff} - \epsilon_a)^{1/2} \tag{3}$$

$$\epsilon_{eff} = (k_x/k_o)^2 \tag{4}$$

Here, p is the transverse wavenumber in the dielectric layer, q is the transverse decay constant in the air region, k_o is the free-space wavenumber, and ϵ_{eff} is the effective dielectric constant which is related to the surface-wave propagation constant, k_x, by (3) and (4). For the waveguide problem, the effective dielectric is the most important quantity to be determined, and it can be obtained from p or q via (2) or (3), respectively, if either one of them is known. It is noted that p and q satisfy the equation:

$$p^2 + q^2 = a^2 \tag{5}$$

with

$$a = k_o(t_f/2)(\epsilon_f - \epsilon_a)^{1/2} \tag{6}$$

Eq.(5) represents a circle of radius a which is commonly refered to as the normalized frequency. In the past, (1) and (5) are considered as a set of coupled nonlinear equations to determine the values for p and q, for a given normalized frequency a. Graphically, the admissive values of p are determined by the intersections a set of modified tangent curves and a circle.

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It was shown that (1) and (5) can be combined to yield a new form of dispersion relation as:

$$a\cos(p) = p \tag{7}$$

where a is the normalized frequency defined by (6). Such a new dispersion relations had been obtained, but they were used only as an auxiliary equation[4]. Because it is simpler in form and offers many advantages over the old one, we shall use it as the basis of the analysis of surface waves, as illustrated below.

Geometrically, the roots of the new dispersion relation, (7), may be interpreted as the intersections of a cosine curve and a straight line. To locate the roots, we have developed a geometrical iteration procedure that are monotonically and rapidly convergent. First of all, the frequency spectrum is divided into two regions, with the normalized dividing frequency given by:

$$a_d = \pi/2 - 1 \tag{8}$$

Specifically, $a < a_d$ defines the low-frequency region, and $a > a_d$ defines the high-frequency region. For the iteration procedure, the initial point or the zeroth-order result is chosen as:

$$p_o = p_a = a, \qquad \text{for } a < p_d \tag{9}$$

$$p_o = p_b = \frac{\pi}{2}\frac{a}{a+1}, \qquad \text{for } a > a_d \tag{10}$$

For a given structure and at a given frequency, the initial point by either one of the last two formulas is very easy to determine. With such an initial point fixed, the first-order result is obtained from the iteration formula, as:

$$p_n = a\,\frac{\cos(p_o) + p_o\sin(p_o)}{1 + a\sin(p_o)} \tag{11}$$

$$f_n = p_n/a \tag{12}$$

The expression for p_n given by (11) is the result of the first iteration and f_n is then determined from (12), if needed. When a more accurate result is desired, the "old" values, p_o and f_o, may be replaced by the "new" values, p_n and f_n, and the process may be repeated until a desired degree of accuracy is achieved. This iteration process will be particularly useful for the purpose of laboratory design using a hand-held calculator.

As an illustration of the usefulness of the present approach, we determine here the frequency range for the operation of a single TE mode. The first higher TE mode begins to propagate at the normalized frequency: $a = \pi/2$. From (12), We have: $p_o = p_a = 0.9598$. Substituting such a value of p_o into (11), we obtain a more accurate value for the dispersion root: $p_n = 0.9341$, as compared to the exact value of $p = 0.9340$. Thus, the frequency range of single mode-operation is given by:

$$0 < p < 0.934 \tag{13}$$

In terms of the effective dielectric constant, (13) translates into:

$$\epsilon_a < \epsilon_{eff} < 0.64\epsilon_f + 0.36\epsilon_a \tag{14}$$

which defines the lower and upper bounds of the effective dielectric constant for the lowest TE mode, with all other higher modes below cutoff. These upper and lower bounds hold generally for the symmetric dielectric waveguide, regardless of the waveguide parameters.

III. Application to waveguide modulator

A waveguide modulator consists of waveguiding layer that is made of an electro-optical or acousto-optical material[6]. The dielectric constant of the layer can be externally controlled to influence the phase of a guided wave. Of the most practical interest is the sensitivity factor of a waveguide modulator, which is defined as:

$$s = \frac{dn_{eff}}{dn_f} = \frac{n_f}{n_{eff}}\,\frac{d\epsilon_{eff}}{d\epsilon_f} \tag{15}$$

The last expression in terms of the product of two factors was obtained by using the square relationship between a dielectric constant and the corresponding index of refraction. Between the two factors above, the former is bounded by:

$$1 < \frac{n_f}{n_{eff}} < \frac{n_f}{n_a} \tag{16}$$

Furthermore, taking the chain differentiations on (1), (4), and (10), we obtain the derivative:

$$g = \frac{d\epsilon_{eff}}{d\epsilon_f} = 1 - \frac{\cos^2(p)}{1 + p\tan(p)} \tag{17}$$

Here, the value of p varies from 0 for a very low frequency to n/2 for a very high frequency. Evidently from the last equation, the factor g is bounded by:

$$0 < g < 1 \tag{18}$$

Combining (15)-(18), we have the sensitivity factor, s, bounded by:

$$0 < s < n_f/n_a \tag{19}$$

These results have been previously demonstrated by means of numerical examples[6]. With the present approach, the sensitivity factor can now be conveniently investigated analytically.

IV. Design of uniform directional couplers

A directional coupler usually consists of two identical constituent waveguides in a uniform environment. The layers have the dielectric constant ϵ_f and the thickness t_f, and the separation between the two waveguides is s. Such

a structure may be viewed as a composite waveguide and the coupling of energy between the two constituent waveguides can be interpreted as a result of interference among the modes supported by the composite structure. In practice, the directional coupler is designed such that each constituent waveguide supports only the lowest mode, and the composite structure supports the two lowest modes with opposite symmetries in field distribution. Thus, the key step to the analysis of the directional coupler is the determination of the propagation constants of the modes supported by the directional coupler as a composite waveguide structure. Here, we derive a set of new, though approximate, dispersion relations which are particularly useful for the study of directional couplers.

The directional coupler as a whole is symmetric with respect to the plane z = 0. Such a structure may be analyzed in terms of two bisected sub-structures, as shown in Fig. 2, together with their network representations. In other words, the modes supported by such a composite waveguide structure have either symmetric or antisymmetric electric-field distribution. For practical applications, the directional coupler is designed such that each basic waveguides suppots only the lowest mode, and the composite structure will support the two lowest modes, one with a symmetric electric-field distribution and the other with an antisymmetric distribution. The coupling of energy between the two basic waveguides forming the directional coupler may be interpretted as the interference of the two modes of the composite structure. Thus, the key step to the analysis of the directional coupler is the determination of the propagatioin constants of the modes supported by the composite structure.

By using the transverse-resonance technique, the dispersion relation is written as:

$$\tan(k_f t) = - \frac{Z_f(X_a + X_{dn})}{Z_f Z_f - X_a X_{dn}} \qquad (20)$$

where Z_f is the wave impedance of the layer, X_a is the wave reactance of the air region, and X_{dn} is the input reactance looking down at the lower interface of the dielectric layer and is given by:

$$X_{dn} = \begin{cases} X_a \tan(k_a s), & \text{for SC bisection} \\ X_a \cot(k_a s), & \text{for OC bisection} \end{cases} \qquad (21)$$

The dispersion relation, (20), determines the effective dielectric constants, ϵ_{eff}, of the directional coupler. With the two different input reactances in (21), (20) will yield the effective dielectric constants for the symmetric and antisymmetric modes. Under the most general condition, the roots of such a dispersion relation can be determined by numerical methods, and the coupling characteristics of the structure can then be analyzed.

For practical design purposes, it is desirable to have explicit formulas for the dispersion roots, particularly, for a nonuniform direction coupler, including the transition regions at the input and output ports. Based on the exact dispersion relation, (20), we present here an approximate method of analysis that will yield the desired reults. The method is algebrac in nature, and is facilitated by a new method of analysis for the basic waveguide consisting of a single dielectric layer in a uniform environment, as described below.

We observe that (20) can be approximated by:

$$\tan(k_f t) = \frac{2 Z_f X_{ave}}{Z_f Z_f - X_{ave}^2} \qquad (22)$$

$$X_{ave} = (X_a + X_{dn})/2 = X_a(1 + u) \qquad (23)$$

$$u = \begin{cases} -\exp(-2k_a s)/[1+\exp(-2k_a s)] \\ \exp(-2k_a s)/[1-\exp(-2k_a s)] \end{cases} \qquad (24)$$

In arriving at (23), only one term of the order u^2 in the denominator had been neglected. In view of (24), such a term is exponentially small, if the separation between the two constituent waveguides, s, is sufficiently large. X_{ave} is the average reactance of the two regions, above and below the waveguide. The approximate dispersion relation, (22), is recognized to be in the same form as that for the simplest dielectric waveguide consisting of a single dielectric layer in a uniform environment. A physical interpretation of such an approximation can be given as follows: The original directional coupler consisting of two identical waveguides possesses the reflection symmetry, but the bisected sub-structures with one waveguide and the boundary condition of electric or magnetic conductuor at the bisection plane on longer possess the symmetry. Since the fields are evanescent in the air regions, the effect of the boundary at the bisection plane is small, if the separation between the two waveguides is sufficiently large. In the derivations from (20) to (22), each bisected structure is replaced by a simpler waveguide consisting of a simple dielectric layer in an unbound uniform environment with a modified reactance which is equal to the average reactance given by (23). In do so, we retain the first order effect of the boundary condition at the bisection plane. It is noted that as far as the dispersion relations are concerned, the only difference between the two bisections is contained in the expressions for u, as given in (24). Thus, the characteristics of the directional coupler can now be analyzed conveniently in terms of such a single quantity.

By using the formula for the double angle of the tangent function, the dispersion relation, (22), can be splitted into two simpler ones, as:

$$\tan(k_f t_f/2) = X_{ave}/Z_f \qquad (25)$$

$$\cot(k_f t_f/2) = -X_{ave}/Z_f \qquad (26)$$

which are recognized as the dispersion relations for short-circuit and open-circuit bisections of

the single dielectric waveguide with the average reactance for the surounding the space. The apperance of X_{ave} accounts for the effect of coupling between the two constituent waveguides. For the case of weak coupling, X_{ave} deviates only slightly from X_a, and the effect of the small deviation can be treated easily by a perturbation procedure. In doing so, we obtained the dispersion roots for the two bisected structures, and the difference in effective dielectric constants of the symmetric and antisymmetric modes of the directional coupler is given by:

$$\epsilon_{eff}^{(s)} - \epsilon_{eff}^{(o)} = \frac{p^2 q^2 \exp(-2qs)}{k_o^2 a^2 (1+u)} \qquad (27)$$

Such an expression can be easily calculated; more importantly, it exhibits the effect of the structure parameters explicitly through the parameters p, q and a, as defined by (2), (3), and (6). For example, p is very small at a high frequency and q at a low frequency. In these two extreme cases, the difference in the effective dielectric constants given by (27) is very small. Therefore, it is expected that an optimum coupling between the two constituent waveguides will occur at an intermediate frequency and this formula provides a simple criterion for practical design purpose.

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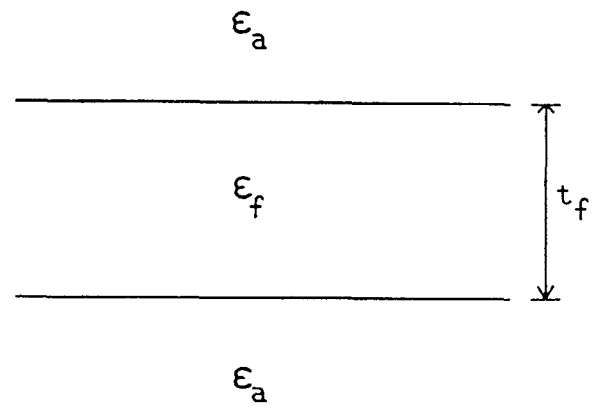

Fig.1. Dielectric layer in uniform environment

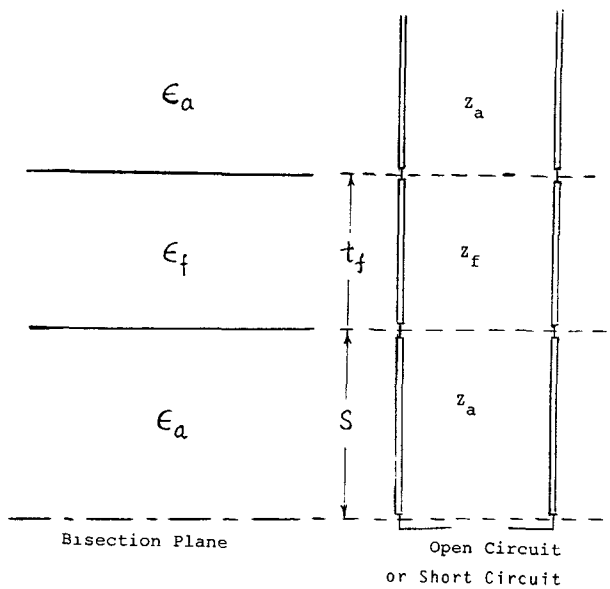

Fig. 2. Bisected directional coupler and its equivalent circuit.